AN EXAMPLE OF TRANSONIC FLOW OF A GAS WITH A SUPERSONIC ZONE, TERMINATED BY A CURVED SHOCK WHICH ENDS INSIDE THE FLOW

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In Ref. [1], Frankl constructed an example of transonic flow, with a supersonic zone that is terminated on the downstream side by a straight shock which ends in the flow. The present work gives an example of the same type, with a curved shock. Our solution is based on the equations of Falkovitch [2], substituting Chaplygin's equation in the vicinity of the sonic speed. Namely,

$$\frac{\partial\varphi}{\partial\theta} = -C\frac{\partial\psi}{\partial\eta}, \qquad \frac{\partial\varphi}{\partial\eta} = C\eta\frac{\partial\psi}{\partial\theta}, \qquad C = \left(\frac{\varkappa+1}{2}\right)^{\frac{1}{\varkappa-1}}(\varkappa+1)^{\frac{1}{3}} \tag{1}$$

where θ is the angle of inclination of the velocity, η is a function of the velocity which was introduced by Frankl [3], ϕ is the velocity potential, ψ is the stream function.

Variation of entropy and thus, of vorticity, behind the shock are neglected, which is permissible near sonic speed, since a jump in entropy is proportional to the cube* of the jump in the normal velocity.

The stream function and the velocity potential separately satisfy the equations

$$\eta\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial\eta^2} = 0, \qquad \frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\frac{\partial\varphi}{\partial\eta}\right) = 0 \tag{2}$$

* Corrected by translator; "square" appears in original.

For the stream function in the subsonic region we shall seek a solution of the form

$$\psi=\psi^{(2)}=\eta^4\left[\psi\left(-\frac{9}{4}\frac{\theta^2}{\eta^3}\right)+\varepsilon g\left(-\frac{9}{4}\frac{\theta^2}{\eta^3}\right)\right] \tag{3}$$

In the $\theta\eta$ plane the shock appears in the form of two curves,

$$\frac{3}{2}\frac{\theta_1}{(-\eta_1)^{3/2}}=k_1,\qquad \frac{3}{2}\frac{\theta_2}{(+\eta_2)^{3/2}}=k_2 \tag{4}$$

in the supersonic half and the subsonic half, respectively, of the plane.

On the characteristic which goes through the origin we have

$$-\frac{9}{4}\frac{\theta^2}{\eta^3}=1 \tag{5}$$

Equation (3) is also valid at points of the $\theta\eta$ plane which are above the characteristic (5). Below this characteristic we have

$$\psi=\psi^{(1)}=\eta^4\left[\alpha f\left(-\frac{9}{4}\frac{\theta^2}{\eta^3}\right)+\beta g\left(-\frac{9}{4}\frac{\theta^2}{\eta^3}\right)\right] \tag{6}$$

On crossing the characteristic (5), the stream function must be continuous, i.e.

$$\alpha f(1)+\beta g(1)=f(1)+\varepsilon g(1) \tag{7}$$

where f and g are hypergeometric functions, as follows:

In the subsonic zone, (8_1)

$$f(z)=-F\left(-1,\,-\tfrac{4}{3},\,\tfrac{1}{2},\,z\right),\quad g(z)=-(-z)^{\frac{1}{2}}\cdot F\left(-\tfrac{1}{2},\,-\tfrac{5}{6},\,\tfrac{3}{2},\,z\right)$$

$$\psi(z)=-\eta^4\left[F\left(-1,\,-\tfrac{4}{3},\,\tfrac{1}{2},\,z\right)+\varepsilon(-z)^{\frac{1}{2}}F\left(-\tfrac{1}{2},\,-\tfrac{5}{6},\,\tfrac{3}{2},\,z\right)\right]$$

Below the characteristic, (8_2)

$$f(z)=-F\left(-1,\,-\tfrac{3}{4},\,\tfrac{1}{2},\,z\right),\quad g(z)=\sqrt{3}z^{\frac{1}{2}}F\left(-\tfrac{1}{2},\,-\tfrac{5}{6},\,\tfrac{3}{2},\,z\right)$$

$$\psi(z)=-\eta^4\left[\alpha F\left(-1,\,-\tfrac{4}{3},\,\tfrac{1}{2},\,z\right)-\beta\sqrt{3}z^{\frac{1}{2}}F\left(-\tfrac{1}{2},\,-\tfrac{5}{6},\,\tfrac{3}{2},\,z\right)\right]$$

Above the characteristic in the supersonic zone

$$f(z)=-F\left(-1,\,-\tfrac{4}{3},\,\tfrac{1}{2},\,z\right) \tag{8_3}$$

$$g(z)=AzF\left(-\tfrac{1}{2},\,-1,\,\tfrac{4}{3},\,\tfrac{1}{z}\right)-B(-z)^{4/3}F\left(-\tfrac{5}{6},\,-\tfrac{4}{3},\,\tfrac{2}{3},\,\tfrac{1}{z}\right)$$

$$\psi(z)=-\eta^4\left\{F\left(-1,\,-\tfrac{4}{3},\,\tfrac{1}{2},\,z\right)-\varepsilon\left[AzF\left(-\tfrac{1}{2},\,-1,\,\tfrac{4}{3},\,\tfrac{1}{z}\right)-\right.\right.$$

$$\left.\left.-B(-z)^{4/3}F\left(-\tfrac{5}{6},\,-\tfrac{4}{3},\,\tfrac{2}{3},\,\tfrac{1}{z}\right)\right]\right\}$$

In these formulas

$$z = -\frac{9}{4}\frac{\theta^3}{\eta^3}, \quad A = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(-\frac{1}{3}\right)}{\Gamma\left(-\frac{5}{6}\right)\Gamma(2)} = 0.5382, \quad B = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{7}{3}\right)\Gamma\left(-\frac{1}{2}\right)} = -0.5625 \tag{8$_4$}$$

The coefficient $-\sqrt{3}$ is chosen so that, along the characteristic, the value of $g(z)$ above the characteristic corresponds to the value below the characteristic. The minus sign in the expression for $f(z)$ is chosen so that the shock will lie to the right of the zero streamline [1]. It is known that the velocity vectors ahead of and behind the shock are related by the relation [4],

$$(\theta_2 - \theta_1)^2 = \frac{1}{2}(\eta_2 - \eta_1)^2(-\eta_1 - \eta_2) \tag{9}$$

The boundary condition on ϕ and ψ along the shock has the form (4),

$$\frac{d\varphi}{d\psi} = \pm C\sqrt{\frac{-\eta_1 - \eta_2}{2}} \tag{10}$$

where C is defined in formula (1).

It is evident that the velocity potential and stream function are continuous across the shock:

$$\varphi^{(1)} = \phi^{(2)} = \varphi, \qquad \psi^{(1)} = \psi^{(2)} = \psi \tag{11}$$

From equations (3), (4) and (11) we find that

$$\alpha f(k_1^2) + \beta g(k_1^2) = k^4[f(-k_2^2) + \varepsilon g(-k_2^2)] \left(k = \frac{\eta_2}{\eta_1}\right) \tag{12}$$

Now, using equations (3), (6) and (8) we write

$$\frac{d\varphi^{(1)}}{d\psi^{(1)}} = \pm C\sqrt{\frac{-\eta_1 - \eta_2}{2}}, \qquad \frac{d\varphi^{(2)}}{d\psi^{(2)}} = \pm C\sqrt{\frac{-\eta_1 - \eta_2}{2}} \tag{13}$$

Next we find ϕ; it can be easily shown that

$$\begin{aligned} \varphi = \varphi^{(1)} &= (-\eta)^{9/2}\left[\alpha f^\circ\left(-\frac{9}{4}\frac{\theta^2}{\eta^3}\right) + \beta g^\circ\left(-\frac{9}{4}\frac{\theta^2}{\eta^3}\right)\right] \quad \text{(below the characteristic)} \\ \varphi = \varphi^{(2)} &= \eta^{9/2}\left[f^\circ\left(-\frac{9}{4}\frac{\theta^2}{\eta^3}\right) + \varepsilon g^\circ\left(-\frac{9}{4}\frac{\theta^2}{\eta^3}\right)\right] \quad \text{(above the characteristic)} \end{aligned} \tag{14}$$

where f^0 and g^0 are hypergeometric functions, as follows:

In the subsonic zone,

$$f^\circ(z) = \frac{8}{3}C(-z)^{1/2}F\left(-\frac{1}{3}, -1, \frac{3}{2}, z\right), \quad g^\circ(z) = -\frac{1}{3}CF\left(-\frac{3}{2}, -\frac{5}{6}, \frac{1}{2}, z\right) \tag{15$_1$}$$

$$\varphi(z) = \eta^{9/2}C\left[\frac{8}{3}(-z)^{1/2}F\left(-\frac{1}{3}, -1, \frac{3}{2}, z\right) - \frac{\varepsilon}{C}F\left(-\frac{3}{2}, -\frac{5}{6}, \frac{1}{2}, z\right)\right]$$

Below the characteristic,

$$f^\circ(z) = -\frac{8}{3}Cz^{1/3}F\left(-\frac{1}{3},\ -1,\ \frac{3}{2},\ z\right),\ g^\circ(z) = \frac{\sqrt{3}}{3}CF\left(-\frac{3}{2},\ -\frac{5}{6},\ \frac{1}{2},\ z\right) \tag{15$_2$}$$

$$\varphi(z) = (-\eta)^{2/3}C\left[-\alpha\frac{8}{3}z^{1/3}F\left(-\frac{1}{3},\ -1,\ \frac{3}{2},\ z\right) + \beta\frac{\sqrt{3}}{3}F\left(-\frac{3}{2},\ -\frac{5}{6},\ \frac{1}{2},z\right)\right]$$

Above the characteristic in the supersonic zone,

$$\begin{aligned} f^\circ(z) &= \frac{8}{3}C(-z)^{1/3}F\left(-\frac{1}{3},\ -1,\ \frac{3}{2},\ z\right) \\ g^\circ(z) &= -\frac{1}{3}C\left[D(-z)^{3/2}F\left(-\frac{3}{2},\ -1,\ \frac{1}{3},\ \frac{1}{z}\right) + \right. \\ &\quad \left. + E(-z)^{5/6}F\left(-\frac{5}{6},\ -\frac{1}{3},\ \frac{5}{3},\ \frac{1}{z}\right)\right] \\ \varphi(z) &= (-\eta)^{2/3}C\left\{\frac{8}{3}z^{1/3}F\left(-\frac{1}{3},\ -1,\ \frac{3}{2},\ z\right) - \right. \\ &\quad \left. -\frac{\varepsilon}{3}\left[Dz^{3/2}F\left(-\frac{3}{2},\ -1,\ \frac{1}{3},\ \frac{1}{z}\right) + Ez^{5/6}\left(-\frac{5}{6},\ -\frac{1}{3},\ \frac{5}{3},\ \frac{1}{z}\right)\right]\right\} \end{aligned} \tag{15$_3$}$$

Here C and z are defined by formulas (1) and (8), while

$$D = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(-\frac{5}{6}\right)\Gamma(2)} = -0.3591,\quad E = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{2}{3}\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(-\frac{3}{2}\right)} = -3.3750 \tag{15$_4$}$$

From equations (3), (6) and (14), and taking into account equations (4) and (12), we obtain from the preceding equations (13),

$$9\sqrt{2}[\alpha f^\circ(k_1^2) + \beta g^\circ(k_1^2)] = 8C\sqrt{1-k}[\alpha f(k_1^2) + \beta g(k_1^2)] \tag{16$_1$}$$

$$9\sqrt{2}[\alpha f^\circ(k_1^2) + \beta g^\circ(k_1^2)] = -8C\sqrt{1-k}[\alpha f(k_1^2) + \beta g(k_1^2)] \tag{16$_2$}$$

$$9\sqrt{2k}[\varphi(-k_2^2) + \varepsilon g^\circ(-k_2^2)] = 8C\sqrt{1-k}[f(-k_2^2) + \varepsilon g(-k_2^2)] \tag{17$_1$}$$

$$9\sqrt{2k}[\varphi(-k_2^2) + \varepsilon g^\circ(-k_2^2)] = -8C\sqrt{1-k}[f(-k_2^2) + \varepsilon g(-k_2^2)] \tag{17$_2$}$$

From equations (7) and (12) we find α and β. Thus

$$\begin{aligned} \alpha &= \frac{k^4[f(-k_2^2) + \varepsilon g(-k_2^2)]g(1) - [f(1) + \varepsilon g(1)]g(k_1^2)}{f(k_1^2)g(1) - f(1)g(k_1^2)} \\ \beta &= \frac{[f(1) + \varepsilon g(1)]f(k_1^2) - k^4[f(-k_2^2) + \varepsilon g(-k_2^2)]f(1)}{f(k_1^2)g(1) - f(1)g(k_1^2)} \end{aligned} \tag{18}$$

Putting (18) into equations (16), we obtain

$$\begin{aligned} &\{9\sqrt{2}[f^\circ(k_1^2)g(1) - f(1)g^\circ(k_1^2)] - 8C\sqrt{1-k}[f(k_1^2)g(1) - \\ &- f(1)g(k_1^2)]\}k^4[f(-k_2^2) + \varepsilon g(-k_2^2)] - [\varphi(k_1^2)g(k_1^2) - \\ &- f(k_1^2)g^\circ(k_1^2)][f(1) + \varepsilon g(1)]9\sqrt{2} = 0 \end{aligned} \tag{19$_1$}$$

$$\begin{aligned} &\{9\sqrt{2}[f^\circ(k_1^2)g(1) - f(1)g^\circ(k_1^2)] + 8C\sqrt{1-k}[f(k_1^2)g(1) - \\ &- f(1)g(k_1^2)]\}k^4[f(-k_2^2) + \varepsilon g(-k_2^2)] - [f^\circ(k_1^2)g(k_1^2) - \\ &- f(k_1^2)g^\circ(k_1^2)][f(1) + \varepsilon g(1)]9\sqrt{2} = 0 \end{aligned} \tag{19$_2$}$$

Now, from equations (4), (9) and (12) we find that

$$\frac{2}{3}\,(k_2 k^{3/2} - k_1) = \sqrt{\frac{1-k}{2}}\,(1+k) \qquad (20_1)$$

or

$$\frac{2}{3}\,(k_1 - k_2 k^{3/2}) = \sqrt{\frac{1-k}{2}}\,(1+k) \qquad (20_2)$$

Thus, we have obtained equations (17), (19) and (20), which make it possible to find the constants α, β, k_1, k_2, k.

From equations (8_1), (8_2) and (15_1), (15_2), expanding in series, we obtain

$$\begin{aligned}
&f(k_1^2) = -(1 + ak_1^2 + \ldots), \qquad f^\circ(k_1^2) = -\frac{8}{3}C(k_1 + mk_1^3 + \ldots)\\
&g(k_1^2) = \sqrt{3}(k_1 + bk_1^3 + \ldots), \qquad g^\circ(k_1^2) = \frac{\sqrt{3}}{3}C(1 + nk_1^2 + \ldots)\\
&f(-k_1^2) = -(1 - ak_2^2 + \ldots), \qquad f^\circ(-k_2^2) = \frac{8}{3}C(k_2 - mk_2^3 + \ldots)\\
&g(-k_2^2) = -(k_2 - bk_2^3 + \ldots), \qquad g^\circ(-k_2^2) = -\frac{1}{3}C(1 - nk_2^2 + \ldots)\\
&a = \frac{8}{3}, \qquad b = \frac{5}{18}, \qquad m = \frac{2}{9}, \qquad n = \frac{5}{2}
\end{aligned} \qquad (21)$$

We also expand the quantities k_1, k_2, k in series:

$$k_1 = \alpha_1\varepsilon + \ldots, \qquad k_2 = \beta_1\varepsilon + \ldots, \qquad k = 1 - \gamma_1^2\varepsilon^2 + \ldots,$$

$$\gamma_1\varepsilon = \sqrt{1-k} + \ldots \qquad (22)$$

Using (21) and (22), equations (17_1), (19_1), (20_2) and (17_2), (19_2), (20_1) take the following form:

$$\begin{aligned}
&(24\sqrt{2}\alpha_1 - 8\gamma - 3\sqrt{6})\,g(1)\,\varepsilon + \ldots = 0\\
&(24\sqrt{2}\beta_1 + 8\gamma_1 - 3\sqrt{2})\,\varepsilon + \ldots = 0\\
&\left[\tfrac{2\sqrt{2}}{3}(\beta_1 - \alpha_1) - 2\gamma_1\right]\varepsilon + \ldots = 0
\end{aligned} \qquad (23_1)$$

$$\begin{aligned}
&(24\sqrt{2}\alpha_1 + 8\gamma_1 - 3\sqrt{6})\,g(1)\,\varepsilon + \ldots = 0\\
&(24\sqrt{2}\beta_1 - 8\gamma_1 - 3\sqrt{2})\,\varepsilon + \ldots = 0\\
&\left[\tfrac{2\sqrt{2}}{3}(\alpha_1 - \beta_1) - 2\gamma_1\right]\varepsilon + \ldots = 0
\end{aligned} \qquad (23_2)$$

From these we obtain

$$\begin{aligned}
&24\sqrt{2}\,\alpha_1 - 8\gamma_1 - 3\sqrt{6} = 0\\
&24\sqrt{2}\,\beta_1 + 8\gamma_1 - 3\sqrt{2} = 0\\
&2(\beta_1 - \alpha_1) - 3\sqrt{2}\,\gamma_1 = 0
\end{aligned} \qquad (24_1)$$

$$\begin{aligned}
&24\sqrt{2}\,\alpha_1 + 8\gamma_1 - 3\sqrt{6} = 0\\
&24\sqrt{2}\,\beta_1 - 8\gamma_1 - 3\sqrt{2} = 0\\
&2(\alpha_1 - \beta_1) - 3\sqrt{2}\,\gamma_1 = 0
\end{aligned} \qquad (24_2)$$

Thus, to determine the constants α_1, β_1, γ_1, it is necessary to use either the system of equations (24_1) or the system (24_2).

In the case of (24_1), we have

$$\alpha_1 = \frac{10\sqrt{3}+1}{88} = 0.2081, \quad \beta_1 = \frac{10+\sqrt{3}}{88} = 0.1333, \quad \gamma_1 = \frac{3\sqrt{2}(1-\sqrt{3})}{88} = -0.0352 \qquad (25)$$

If the system of equations (24_2) is used, the other value of γ_1 is obtained. It turns out that for $\epsilon < 0$ it is necessary to take the first value, i.e. the system (24_1), and for $\epsilon > 0$, the second value, i.e. the system (24_2); but $\gamma_1 \epsilon$ is positive by definition.

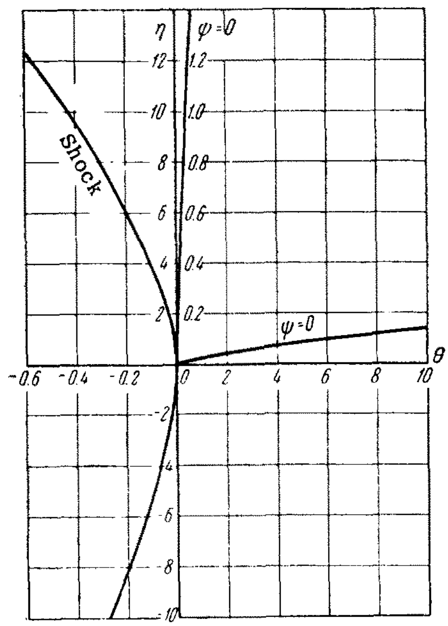

Fig. 1.

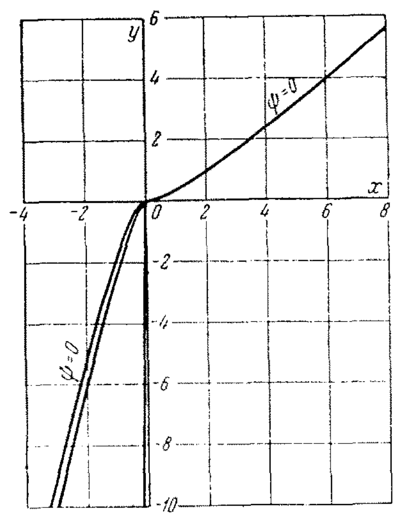

Fig. 2.

Since, in what follows, we take $\epsilon < 0$, for reasons that appear later, we use the solution of the first system. Then, with the values from (25), equations (22) give

$$k_1 = 0.208\,\varepsilon + \ldots, \qquad k_2 = 0.1333\,\varepsilon + \ldots, \qquad k = 1 - 0.0012\,\varepsilon^2 + \ldots$$

In what follows, we will assume $\epsilon < 0$, and specifically will take $\epsilon = -0.1$. In this case, $\theta > 0$ with $\eta = 0$, we find $\psi < 0$, from which it follows that the approaching streamline which passes through the end of the shock lies entirely in the subsonic region.

Our problem then is the construction of this particular example. We have

$$k_1 = -0.0208, \qquad k_2 = -0.0133, \qquad k = 0.9999 \tag{26}$$

From equations (4), together with (26), we have

$$\theta_1 = -0.0138\,(-\eta_1)^{3/2}, \quad \theta_2 = -0.0088\,\eta_2^{3/2} \tag{27}$$

This curve is shown in Fig. 1, in the θ,η plane. From equations (27) and (12) we obtain $\theta_2/\theta_1 = 0.6376\ k^{3/2}$.

Taking the value of k from equation (26),

$$\theta_2 = 0.6375\ \theta_1 \tag{28}$$

In equation (1), $\kappa = 1.4$ for air; then $C = 2.111$. Using (26) we obtain from equations (21), to a first approximation, (29)

$$f(k_1^2) = -1, \qquad g(k_1^2) = -0.0360, \qquad f^\circ(k_1^2) = 0.1171, \qquad g^\circ(k_1^2) = 1.2188$$
$$f(-k_2^2) = -1, \qquad g(-k_2^2) = 0.0133, \qquad f^\circ(-k_2^2) = -0.0748, \qquad g^\circ(-k_2^2) = -0.7036$$

From equations (18) and (29) we obtain

$$\alpha = 1.0034, \qquad \beta = -0.0945 \tag{30}$$

Next we find the relation of x and y to η, along the shock:

$$\psi = \psi^{(1)} = \psi^{(2)} = -1.0013\,\eta^4, \qquad \varphi = \varphi^{(1)} = \varphi^{(2)} = -0.0045\,\eta^{5/2} \tag{31}$$

Now from Chaplygin's formulas, we have

$$dx = \frac{\cos\theta}{w}\,d\varphi - \frac{\rho_0}{\rho w}\sin\theta\,d\psi, \qquad dy = \frac{\rho_0}{\rho w}\cos\theta\,d\psi + \frac{\sin\theta}{w}\,d\varphi$$

We find that

$$dx = \left[\frac{1}{a^*} + \frac{d}{d\eta}\left(\frac{1}{w}\right)_{|\eta=0}\eta + \ldots\right]d\varphi - \left[\frac{\rho_0}{\rho^* a^*} + \ldots\right](\theta + \ldots)\,d\psi$$

$$dy = \left[\frac{\rho_0}{\rho^* a^*} + \frac{d}{d\eta}\left(\frac{\rho_0}{\rho w}\right)_{|\eta=0}\eta + \frac{1}{2}\frac{d^2}{d\eta^2}\left(\frac{\rho_0}{\rho w}\right)_{|\eta=0}\eta^2 + \ldots\right]d\psi + (\theta + \ldots)\left(\frac{1}{a^*} + \ldots\right)d\varphi$$

or

$$a^*\,dx = d\varphi + [(\varkappa+1)^{-1/3}\eta + \ldots]\,d\varphi - \left(\frac{\varkappa+1}{2}\right)^{\frac{1}{\varkappa-1}}(0 + \ldots)\,d\psi$$

$$a^*\,dy = \left(\frac{\varkappa+1}{2}\right)^{\frac{1}{\varkappa-1}}\left[1 + \frac{(\varkappa-1)^{1/3}}{2}\eta^2 + \ldots\right]d\psi + \left(\frac{\varkappa+1}{2}\right)^{\frac{1}{\varkappa-1}}(\varkappa+1)^{1/3}\,0\,d\varphi$$

Integrating both parts, we obtain

$$a^* x = \varphi + (\varkappa+1)^{-\frac{1}{3}}\int \eta\,d\varphi - \left(\frac{\varkappa+1}{2}\right)^{\frac{1}{\varkappa-1}}\int 0\,d\psi + \ldots$$

$$a^* y = \left(\frac{\varkappa+1}{2}\right)^{\frac{1}{\varkappa-1}}\psi + \left(\frac{\varkappa+1}{2}\right)^{\frac{1}{\varkappa-1}}\frac{(\varkappa+1)^{1/3}}{2}\int \eta^2\,d\psi + \left(\frac{\varkappa+1}{2}\right)^{\frac{1}{\varkappa-1}}(\varkappa+1)^{1/3}\int \theta\,d\varphi \tag{32}$$

From here (with $\theta = -0.0088\,\eta^{3/2}$, and $\kappa = 1.4$ for air), together with equation (31), we obtain

$$a^* x = -0.0045\,\eta^{9/2} - 0.0129\,\eta^{11/2}, \qquad a^* y = -1.5791\,\eta^4 - 0.7045\,\eta^6 \tag{33}$$

Thus, in this given case, we obtain the equations of the shock, which, looking from the end of the shock, is directed upstream.

Next, we will find x and y for $\eta = 0$ (i.e. the sonic line); for this, we express ψ and ϕ in terms of θ. We know that

$$\psi = -\eta^4 \left[F\left(-1, -\frac{4}{3}, \frac{1}{2}, z\right] + \varepsilon(-z)^{1/2} F\left(-\frac{1}{2}, -\frac{5}{6}, \frac{3}{2}, z\right)\right]$$

$$\varphi = \frac{C}{3}\eta^{9/2}\left[8(-z)^{1/3} F\left(-\frac{1}{3}, -1, \frac{3}{2}, z\right) + \varepsilon F\left(-\frac{3}{2}, -\frac{5}{6}, \frac{1}{2}, z\right)\right]$$

For $\eta = 0$

$$\psi = -0.1658\,\theta^{8/3}, \qquad a^* x = \varphi - 1.5777 \int \theta\, d\psi$$

$$\varphi = -4.3075\,\theta^3, \qquad a^* y = 1.5777\,\psi + 2.1111 \int \theta\, d\varphi$$

or

$$a^* x = -4.3070\,\theta^3 + 0.1901\,\theta^{11/3}, \qquad a^* y = -0.1829\,\theta^{8/3} - 6.8190\,\theta^4 \tag{34}$$

Next we find x and y on $\psi = 0$ (i.e. the zero streamline). We know that

$$\psi = -\eta^4\left[F\left(-1\ -\frac{4}{3}, \frac{1}{2}, z\right) + \varepsilon(-z)^{1/2} F\left(-\frac{1}{2}, -\frac{5}{6}, \frac{3}{2}, z\right)\right] =$$
$$= -\eta^4\left\{\left(1 + \frac{8}{3}z\right) + \varepsilon\left[(-z)\,AF\left(-\frac{1}{2}, -1, \frac{4}{3}, \frac{1}{2}, \frac{1}{z}\right) +\right.\right.$$
$$\left.\left. + (-z)^{4/3} BF\left(-\frac{5}{6}, -\frac{4}{3}, \frac{2}{3}, \frac{1}{z}\right)\right]\right\} = -\left(\frac{9}{4}\right)^{4/3}\theta^{8/3}\left\{(-z)^{-4/3}\left(1 + \frac{8}{3}z\right) +\right.$$
$$+ \varepsilon\left[(-z)^{-1/3} AF_1\left(\frac{1}{z}\right) + BF_2\left(\frac{1}{z}\right)\right]\Big\} = -\left(\frac{9}{4}\right)^{4/3}\theta^{8/3}\left\{\tau\left(\tau - \frac{8}{3}\right) +\right.$$
$$\left. + \varepsilon\left[\tau A\left(1 - \frac{8}{3}\tau^3 + \dots\right) + B\cdot F_2(-\tau^3 + \dots)\right]\right\} = -\left(\frac{9}{4}\right)^{4/3}\theta^{8/3}\{h_1(\tau) + \varepsilon h_2(\tau)\}$$

Here

$$h_1(\tau) = \tau\left(\tau^3 - \frac{8}{3}\right), \qquad h_1'(0) = -\frac{8}{3}$$

$$h_2(\tau) = A\tau\left(1 - \frac{8}{3}\tau^3\right) + BF_2(-\tau^3)$$

$$h_2'(0) = B = -\frac{9}{16} = -0.5625, \qquad \tau = \left(\frac{4}{9}\frac{\eta^3}{\theta^2}\right)^{1/3} = (-z)^{-\frac{1}{3}}$$

for which A and B are given by formulas (8).

Let $\psi = 0$ for $\tau = \tau'$ and $\tau = \tau''$ (i.e. for $z = z'$ and $z = z''$); then

$$h(\tau) = h_1(\tau) + \varepsilon h_2(\tau) = 0$$

It is easily seen that

$$h_1(\tau) = 0 \quad \text{for} \quad \tau = 0 \quad \text{and} \quad \tau = (8/3)^{1/3}$$

Thus, $h_1(0) = h_1[(8/3)^{1/3}] = 0$. Let $\tau = \tau'$; then

$$h(\tau') = h_1(\tau) + \varepsilon h_2(\tau) = 0$$

Expanding $h_1(\tau')$ in a Taylor series we obtain

$$h_1(\tau') = h_1(0) + \tau' h_1'(0) + \ldots = \tau' h_1'(0) + \ldots$$

It follows that

$$\tau' h_1'(0) + \varepsilon h_2(0) = 0, \qquad \tau' = -\varepsilon \frac{h_2(0)}{h_1(0)} = \frac{3}{8} B\varepsilon = -\frac{3}{8}\frac{9}{16}\varepsilon$$

or

$$\tau' = 0.0211, \qquad z' = -\frac{1}{\tau'^3} = 106\,382.9780 \tag{35}$$

This value, $z = z'$, corresponds to the front part of the streamline. The other value, $z = z''$, corresponding to the rear part (downstream from the end of the shock) is close to the root of the equation $f(z) = 0$ ($z = -3/8$).

Expanding in series we have, to a first approximation,

$$0 = (z'' - z_0) f'(z_0) + \varepsilon g(z_0)$$

$$z'' - z_0 = -\varepsilon \frac{g(z_0)}{f'(z_0)} = -\frac{3}{8} \varepsilon g(z_0)$$

since $f'(z_0) = 8/3$. Thus

$$z'' = \frac{3}{8}\left[1 - \left(\frac{3}{8}\right)^{1/2} \varepsilon F\left(-\frac{1}{2}, -\frac{5}{6}, \frac{3}{2}, -\frac{8}{3}\right)\right] = -0.3545 \tag{36}$$

In this manner we obtain $z' = -106382.9780$, $z'' = -0.3545$. It follows that the equations

$$\eta = \left(-\frac{9}{4}\frac{\theta^2}{z'}\right)^{1/3} = 0.0276\,\theta^{2/3}$$

$$\eta = \left(-\frac{9}{4}\frac{\theta^2}{z''}\right) = 1.8517\,\theta^{2/3} \tag{37}$$

define the zero streamline in the $\theta\eta$ plane; it is shown graphically in Fig. 1.

From equation (32), for $\psi = 0$, we obtain

$$a^* x = \varphi + (\varkappa + 1)^{-1/3} \int \eta \, d\varphi, \quad a^* y = \left(\frac{\varkappa + 1}{2}\right)^{\frac{1}{\varkappa - 1}} (\varkappa + 1)^{1/3} \int \theta \, d\varphi \tag{38}$$

Now, using the value $z = z' = -106382.9780$, we calculate ϕ. Corresponding to (15),

$$\varphi = \eta^{3/2} \frac{C}{3}\left[8(-z)^{1/2} F\left(-\frac{1}{3}, -1, \frac{3}{2}, z\right) - \varepsilon F\left(-\frac{3}{2}, -\frac{5}{6}, \frac{1}{2}, z\right)\right] =$$

$$= C(4\eta^3\theta - 2\theta^3) + \frac{0.1}{3} C\left[\left(\frac{9}{4}\right)^{3/2} D\theta^3 F\left(-\frac{3}{2}, -1, \frac{1}{3}, \frac{1}{z}\right) +\right.$$

$$+E\left(\frac{3}{2}\theta\right)^{5/3}\eta^2F\left(-\frac{5}{6},\,-\frac{1}{3},\,\frac{5}{3},\,\frac{1}{z}\right)\Big]=$$

$$=2.1111\,\theta^3\left\{4\,\frac{\eta^3}{\theta^2}-2+\frac{0.1}{3}\left[\left(\frac{9}{4}\right)^{3/2}D\left(1-\frac{3}{2}\,3\,\frac{1}{z}\right)+\right.\right.$$

$$\left.\left.+E\left(\frac{3}{2}\right)^{5/3}\frac{\eta^2}{\theta^{4/3}}\,F\left(-\frac{5}{6},\,-\frac{1}{3},\,\frac{5}{3},\,\frac{1}{z}\right)\right]\right\}=$$

$$=2.1111\,\theta^3\left\{9\tau^3-2+\frac{0.1}{3}\left[\left(\frac{9}{4}\right)^{3/2}D\left(1+\frac{9}{2}\,\tau^3\right)+\right.\right.$$

$$\left.\left.+E\left(\frac{3}{2}\right)^3\tau^2F\left(-\frac{5}{6},\,-\frac{1}{3},\,\frac{5}{3},\,-\tau^3\right)\right]\right\}$$

Taking $r = r' = 0.0211$, we obtain

$$\varphi=-4.1166\,\theta^3 \tag{39}$$

From the other side,

$$\varphi=\eta^{9/2}\,\frac{C}{3}\left[8\,(-z)^{1/2}F\left(-\frac{1}{3},\,-1,\,\frac{3}{2},\,z\right)-\varepsilon F\left(-\frac{3}{2},\,-\frac{5}{6},\,\frac{1}{2},\,z\right)\right]$$

so that, for $z = z'' = -0.3545$ we obtain

$$\varphi=3.0736\,\eta^{9/2} \tag{40}$$

From equations (39), (40) and (38), taking into account (36), we obtain

$$a^*x=-4.1166\,\theta^3-0.0694\,\theta^{11/3},\qquad a^*y=-6.5179\,\theta^4 \tag{41}$$

$$a^*x=49.1668\,\theta^3+55.6299\,\theta^{11/3},\qquad a^*y=77.8468\,\theta^4 \tag{42}$$

Table 1 gives the coordinates of the shock and the sonic line in the xy plane, and Table 2 gives the coordinates of the zero streamline.

Table 1

Shock			Sonic line		
η	$-x$	$-y$	θ	$-x$	$-y$
0.50	0.0005	0.1096	0.50	0.5235	0.4549
0.75	0.0038	0.6249	0.75	1.7500	2.2424
1.00	0.0174	2.2836	1.00	4.1169	7.0019
1.25	0.0564	6.5426	1.25	7.9813	16.9794
1.50	0.1478	16.0186	1.50	13.7956	35.0602
1.75	0.3359	35.0454	1.75	21.6036	64.7680
2.20	0.6854	70.2536	2.00	32.0425	110.2650
2.25	1.2900	131.8769	2.25	45.3415	176.3532
2.50	2.2701	233.6806	2.50	61.8255	268.4728
2.75	3.7904	395.0176	2.75	81.8124	392.7062
3.00	6.0610	641.4876	3.00	105.6150	555.8628

Table 2

Equation (41)			Equation (42)	
θ	x	y	x	y
0.10	— 0.0041	— 0.0006	0.0602	0.0077
0.25	— 0.0646	— 0.0254	1.1119	0.3036
0.50	— 0.5199	— 0.4074	10.5238	4.8654
0.75	— 1.7604	— 2.1622	40.1088	24.6307
1.00	— 4.1860	— 6.5179	104.7967	77.8468
1.25	— 8.1973	— 15.9128	222.0627	109.0551
1.50	—14.2003	— 32.9968	411.8943	394.0994
1.75	—22.6025	— 61.1307	696.4052	730.1173
2.00	—33.8139	—104.2864	1099.6116	1245.5488
2.25	—48.2478	—167.0466	1648.0265	1995.1278
2.50	—66.3192	—254.6054	2369.3264	3040.8906
2.75	—88.4448	—372.7710	3293.2022	4452.2064

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